Parameter Robust Linear-Quadratic-Gaussian Design Synthesis with Flexible Structure Control Applications

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An asymptotic linear quadratic Gaussian design synthesis technique is described that explicitly includes a class of structural plant uncertainties by viewing plant parameter variations as an internal feedback loop. A direct structural relationship between this class of parameter uncertainties and the weighting matrices in the design of the LQG compensator is exploited by an asymptotic procedure in which either the regulator or the filter becomes insensitive to parameter variations. This asymptotic approach represents a generalization of the linear-quadratic-Gaussian/loop transfer recovery (LQG/LTR) technique. Controllers designed by this new LQG method and LQG/LTR are compared by application to a mass-spring-damper system, which approximates the dynamics of flexible structures. For both colocated and noncolocated sensor and actuator configurations, the LQG/LTR design is extremely sensitive to parameter variations, whereas the new LQG design allows considerably improved parameter robustness.

I. Introduction

PLANT uncertainties such as parameter variation and unmodeled dynamics (or truncated higher-order dynamics) are important considerations in the design of robust controllers for flexible space structures. Stability problems stemming from unmodeled dynamics, usually referred to as the spillover phenomena, are relatively well understood.[1] Many design methods have been proposed to handle such difficulties (see Ref. 2 and references therein). However, the robustness issue associated with active control under parameter uncertainty has not yet drawn much attention, partly because few synthesis techniques are tailored to parameter uncertainty. Indeed, the inclusion of structural plant parameter variations in compensator design has been a perennial control problem. In this paper, we apply to the active control of flexible structures a state-space domain synthesis technique,[3] called here the parameter robust linear quadratic Gaussian (PRLQG) synthesis technique.

In general, structural vibrations for the attitude control of conventional spacecraft have not been a critical problem because of low bandwidth requirements (typically 0.1 Hz or lower) and relatively small and rigid structures. Therefore, the flexible (or vibrational) modes need not be actively controlled. Because of higher bandwidth requirements and lower flexible mode frequencies, however, many future space applications will require active suppression of flexible modes. Active control (or active vibration suppression) results in strong interaction between the poles and zeros of the controller and those of the plant within the bandwidth. Hence, any variations in the locations of the plant poles or zeros as a result of parameter variations will affect the way they interact, possibly inducing instability. In fact, this possible instability caused by parameter variation can be a critical issue in the active control of large space structures because accurate prelaunch identification of system parameters, such as damping ratio and stiffness, may be inaccurate in the space environment. Mass and inertia properties may also be subject to changes, depending on configuration growth and operational modes. Onboard system identification does not completely solve the parameter robustness problem because biases in parameter identification resulting from external noise or unmodeled dynamics always exist.[4]

Owing to low damping in space, the plant poles and zeros are located very close to the imaginary axis. Therefore, even for small parameter variations, gain and phase of the plant can be very uncertain around the nominal frequencies of these poles and zeros. Hence, the frequency-domain, singular-value uncertainty models of input-output maps, such as those used in Ref. 5, will show peaks and notches at these frequencies. In general, since it is prohibitive to find analytic expressions of nonconservative uncertainty bounds, the various modern frequency-domain robust synthesis techniques[6-9] based on these singular-value uncertainty models may produce controllers too conservative for the active control of flexible structures. Conservative norm bounds can be used only to produce low-bandwidth design. In fact, recent studies in control of large flexible space structures using linear-quadratic-Gaussian/loop transfer recovery (LQG/LTR)[10,11] have avoided including parameter variations where a potential robustness problem exists.

Gain-phase stabilization, such as notch compensation, which is a classical technique for handling flexible modes, can be used to accommodate the mode uncertainties. However, the robust gain-phase stabilization becomes very complicated as more flexible modes are included in the plant model, even for the single-input, single-output case. On the other hand, the convention LQG control is devoid of stability robustness because it produces near pole-zero cancellation, which is very sensitive to parameter variations.[12] Therefore, parameter variations can destabilize not only plant modes but also the LQG controller modes placed to cancel plant zeros. In spite of its guaranteed stability margins, the LQG/LTR procedure shares this difficulty. Indeed, the sheer numerical values of stability margins can be deceptive, as will be demonstrated in this paper.

The PRLQG synthesis employed in this paper is based on a state-space model of parameter variations.[3] State-space models preserve the structure of parameter variations better than frequency-domain models, which are better suited for representing unmodeled dynamics. The key idea of the state-spaced model for the PRLQG synthesis is that a parameter variation can be equivalently expressed as an internal feedback loop (IFL). The PRLQG synthesis exploits the structural relationship between the optimal LQG weighting matrices and the

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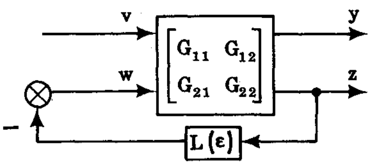


Fig. 1 TITO representation of perturbed systems.

state-space parameter variation model to provide an asymptotic parameter robustness procedure.

In Sec. II, the IFL representation and, in Sec. III, the PRLQG synthesis procedure are briefly explained. In Sec. IV, the performance of controllers designed by PRLQG and LQG/LTR synthesis techniques are compared by application to a mass-spring-damper system that approximates the dynamics of a flexible structure. Both colocated and noncolocated sensors and actuators configurations are considered.

II. Internal Feedback Loop Representation

In this section, a parameter variation is shown to be equivalently represented as an internal feedback loop (IFL) via the input-output representation. This representation of parameter variation is plant-independent in the sense that there are no restrictions on the number of CRHP poles or zeros of the perturbed plant as there are in frequency-domain modeling.[5,13] Furthermore, the structure of parameter variation is easily embedded in a state-space model by the use of the IFL representation.

Let (A,B,C) and $(\hat{A},\hat{B},\hat{C})$ represent the nominal plant (design model) and perturbed plant (real plant), respectively. In the following discussion, we assume that only the state matrix A is subject to variation. Parameter variations in the matrices B and C can be embedded in an augmented state matrix as shown in Ref. 3.

Suppose that $\Delta A \equiv \hat{A} - A$ is a function of p parameters, $\epsilon = \{\epsilon_1,\epsilon_2,...\epsilon_p\}$, and can be decomposed as

$$\Delta A(\epsilon) = -ML(\epsilon)N$$

where M and N are constant matrices and $L(\epsilon)$ is a matrix function of $\epsilon = \{\epsilon_1,\epsilon_2,...\epsilon_p\}$. The parameters $\epsilon = \{\epsilon_1,\epsilon_2,...\epsilon_p\}$ can be viewed as the independent parameters associated with the parameter variation ΔA. The decomposition just described is called input-output (I/O) decomposition of ΔA. Given an I/O decomposition of $\Delta A(\epsilon)$, the perturbed plant can be written as

$$\dot{x} = Ax + Bu + Mw \qquad y = Cx \qquad z = Nx \qquad w = -L(\epsilon)z$$

where z and w are introduced as an auxiliary output and input, respectively, for the internal feedback loop with a gain $L(\epsilon)$. Then,

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} C\phi B & C\phi M \\ N\phi B & N\phi M \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix}$$

where $\phi = (sI - A)^{-1}$.

Suppose that a feedback compensator $K(s)$ is chosen

$$u(s) = v(s) - K(s)y(s)$$

where v is the command or reference input. The perturbed closed-loop system is then depicted in Fig. 1 as a two-input, two-output system given as

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}$$
$$= \begin{bmatrix} (I+g_{11}K)^{-1}g_{11}K & (I+g_{11}K)^{-1}g_{12} \\ g_{21}(I+Kg_{11})^{-1}K & g_{22} - g_{21}K(I+g_{11}K)^{-1}g_{12} \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}$$

It is straightforward to show that the closed-loop system is stable if

$$\det[I + \alpha L(\epsilon)G_{22}(j\omega)] \neq 0$$

for all $\alpha \in [0,1]$ and $\omega \in \mathbb{R}$ (Ref. 3). This condition explicitly shows the dependence of stability robustness on G_{22}, which reflects the structure of the parameter variation via the matrices M and N. Hence G_{22} will be called the *robustness matrix function* in this paper. By applying elementary singular-valve inequalities, we also obtain a sufficiency condition in terms of the infinity norm of G_{22}; i.e., the closed loop is stable if

$$\|G_{22}\|_\infty = \sup_\omega \bar{\sigma}[G_{22}(j\omega)] < 1/\bar{\sigma}[L(\epsilon)]$$

Thus, the infinity norm of G_{22} specifies the maximum tolerable magnitude of $L(\epsilon)$. For the numerical examples of Sec. IV, $\bar{\sigma}[G_{22}(j\omega)]$ will be used to check the parameter robustness. In the next section, it is shown that, by an asymptotic procedure, the robustness matrix function can be made independent of either the filter or controller gains. Therefore, parameter robustness of the LQ regulator or the Kalman filter, respectively, is recovered.

III. Parameter Robust LQG Synthesis

The PRLQG synthesis technique is an asymptotic procedure based on reducing the norms associated with G_{22}, where G_{22} is constructed from the input and output matrices M and N formed for the I/O decomposition of the system parameter variation model. In Ref. 3, it is shown that if the gains of the full-state observer or controller can be constructed in a particular way, asymptotically the robustness matrix function will be dependent only on the controller or filter gains, respectively. That is, the filter or controller, respectively, is mode-intensitive to the parameter variations. To construct the filter and controller gains, the LQG compensator is used. In particular, the LQG asymptotic procedure is developed by using the input and output matrices M and N to form the LQG weighting matrices. To motivate explicitly how these weightings are formed and related to the robustness matrix function, the LQG problem is presented with the following special structure: Find the control u, which minimizes the cost

$$J_{lqg} = \lim_{\tau\to\infty} \frac{1}{\tau} E\left[\int_0^\tau (y_c^T y_c + \rho^2 u^T u)\, dt\right]$$

subject to

$$\dot{x} = Ax + Bu + R\xi \qquad y = Cx + \mu\eta \qquad y_c = Hx$$

where $E[\cdot]$ denotes the expectation operation, $\xi \in \mathbb{R}^q$ and $\eta \in \mathbb{R}^l$ are vector Gaussian white-noise precesses with identity as the spectral density of appropriate dimension, and ρ and μ are scalars. Note that y represents the system measurements and $y_c \in \mathbb{R}^p$ denotes a measurement of the state as observed by the performance index. The solution to this problem is composed of a linear filter

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x})$$

in cascade with a linear controller

$$u = -K_c\hat{x}$$

where the optimal gain matrices for the filter and controller are, respectively,

$$K_c = \frac{1}{\rho^2}B^T\Gamma, \quad K_f = \frac{1}{\mu^2}\Sigma C^T$$

where Γ and Σ are the positive definite solutions to the algebraic Riccati equations

$$\Gamma A + A^T\Gamma + H^T H - \frac{1}{\rho^2}\Gamma B B^T \Gamma = 0$$

$$\Sigma A^T + A\Sigma + RR^T - \frac{1}{\mu^2}\Sigma C^T C \Sigma = 0$$

assuming that (A,B,H) and (A,C,R) are minimal realizations. In the preceding expressions, $H^T H$ and $\rho^2 I$ are the weightings on state and control, respectively, while RR^T and $\mu^2 I$ are, respectively, the power spectral density matrices of process noise and measurement noise. In terms of the transformed variables, the LQG compensator is obtained as

$$u(s) = -K(s)y(s)$$

where

$$K(s) = K_c(sI - A + BK_c + K_f C)^{-1} K_f$$

Also, using the state equation,

$$\begin{bmatrix} y \\ y_c \end{bmatrix} = \begin{bmatrix} C\phi B & C\phi R & \mu I \\ H\phi B & H\phi R & 0 \end{bmatrix} \begin{bmatrix} u \\ \xi \\ \eta \end{bmatrix}$$

By algebraic manipulations, the elements of the quadratic terms in the cost can be represented in terms of the system noise.

$$\begin{bmatrix} y_c \\ \rho u \end{bmatrix} = P(s) \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

where the coefficient is a function of the optimal gains

$$P(s) = \begin{bmatrix} H\phi BK(I+GK)^{-1}C\phi R & -\mu H\phi BK(I+GK)^{-1} \\ -\rho K(I+GK)^{-1}C\phi R & -\mu\rho K(I+GK)^{-1} \end{bmatrix}$$

and $G = C\phi B$. Applying Parseval's theorem, we obtain

$$J_{lqg} - \frac{1}{\pi}\int_0^\infty tr[P(j\omega)^H P(j\omega)]\, d\omega = \frac{1}{\pi}\, \|P(j\omega)\|_2$$

where the subscript $(\cdot)^H$ denotes the complex conjugate transpose.

The next step is to show that the robustness function G_{22} can be minimized in the two-norm sense via an LQG algorithm. Let

$$J_R = \frac{1}{\pi}\, \|G_{22}(s)\|_2 = \frac{1}{\pi}\, \|N\phi M - N\phi BQC\phi M\|_2$$

where $Q = K(I + GK)^{-1}$. Suppose that H and R are selected as $N = P_c H$ and $H = RP_f$, where P_c and P_f are any compatible matrices. Then, it is easy to see that as $\rho \to 0$ and $\mu \to 0$, $J_R \to 0$ if $J_{lqg} \to 0$. It is noted that both the Kalman filter and the LQ regulator are asymptotic, resulting in infinite K_c and K_f. Since the infinity norm of G_{22}, rather than its two norm, defines a sufficient condition for stability robustness, the two-norm optimization (or H^2 optimization) does not necessarily optimize the robustness to ΔA. However, a sufficiency condition for the robustness optimization can be found by first considering either an asymptotic filter or asymptotic regulator.

Suppose that $C \in \mathbb{R}^{l\times n}$ and $R \in \mathbb{R}^{n\times l}$ are full rank. Then, as $\mu \to 0$ with a finite ρ, $R = \lim_{\mu\to 0} \mu K_f$ if $C(sI - A)^{-1}R$ has no zero in the closed right-half plane (CRHP).[14] Then, according to Ref. 3,

$$G_{22}(j\omega) \to N(j\omega I - A + BK_c)^{-1}M$$

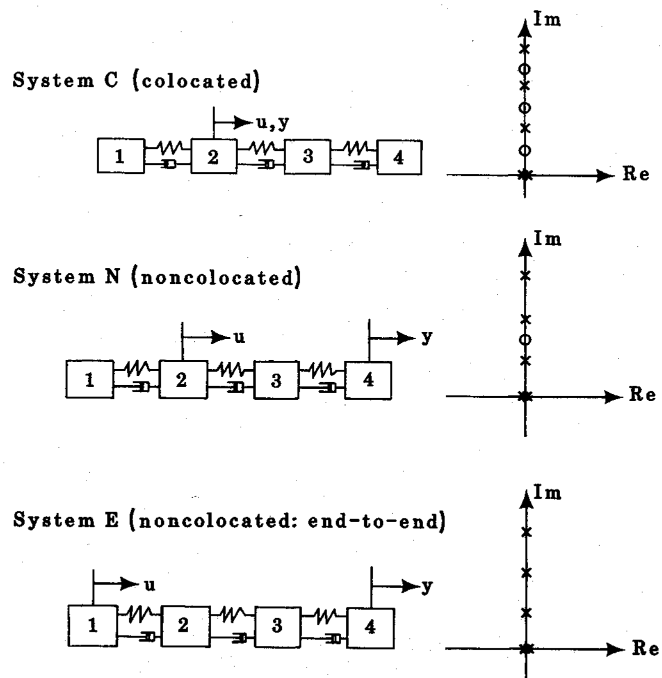


Fig. 2 Spring-mass-damper systems.

for all ω if a P_f exists such that $M = RP_f$. In fact, we can easily show that $N(sI - A + BK_c)^{-1}M$ is the robustness matrix function of the LQ regulator, given that $\Delta A = -ML(\epsilon)N$. Hence, we see that an asymptotic filter obtained via a Kalman filter design recovers the parameter robustness matrix function of an LQ regulator. An exact dual of this robustness property exists for the asymptotic LQ regulator; i.e., the robustness of a Kalman filter is recovered by an asymptotic LQ regulator if $N(sI - A)^{-1}B$ is minimum-phase (no CRHP zeros) and H is chosen in such a way that $N = P_c H$. Furthermore, both an asymptotic filter and asymptotic regulator can be employed such that $G_{22} \to 0$, if all the conditions for the asymptotic filter and regulator are met. In other words, the two-norm optimization of G_{22} via the LQG synthesis can be used to optimize the parameter robustness in the infinity-norm sense.

If rank $(M) >$ rank $(C) =$ rank (R), then we cannot find a P_f such that $M = RP_f$. This implies that the number of outputs imposes a restriction on the rank of M in the PRLQG synthesis. Similarly, the number of inputs restricts the rank of N. However, as exemplified in Sec. IV, the asymptotic LQG procedure can be used as a robustness compromise between parameter variations, even if the rank conditions are not satisfied.

It is important to note that a direct structural relationship exists between the parameter variations and the optimal weighting matrices; i.e., $M = RP_f$ and $N = P_c H$. Using this simple relationship, the PRLQG synthesis forces the asymptotic part of the LQG control system to be insensitive to the parameter variation described by M and N. The robustness of the overall system is then determined by the nonasymptotic part. The simultaneous asymptotic procedure ($\rho \to 0$ and $\mu \to 0$) can be used to improve robustness further.

The use of a white-noise process for modeling parameter errors is well known in the standard LQG design synthesis.[15] These theoretical results justify this empirical procedure by relating the LQG problem to the robustness matrix function, and thereby to the internal feedback loop representation. It is also noted that the parameter robust LQG procedure described earlier is a generalization of the LQG/LTR method in which the asymptotic gain using $H = C$ recovers the loop-transfer function at the output (sensitivity recovery) or the asymptotic filter gain $R = B$ recovers the loop-transfer functions at the input (robustness recovery).[6]

IV. Numerical Examples

In this computational experiment, three input-output configurations have been considered, as shown in Fig. 2. For convenience, each system is named: system N, system C, and system E. System C is colocated, system N noncolocated, and system E noncolocated but with an end-to-end configuration. It is noted that this mass-spring-damper system is equivalent in its mathematical representation to the disk-torsion bar problem considered by Cannon and Rosenthal.[16]

Only SISO configurations are studied. However, it is more difficult to design controllers for SISO systems for parameter robustness than for MIMO systems because the number of the inputs and outputs restrict the number of parameter variations allowed for asymptotic robustness. A single parameter variation and multiple parameter variations will be considered for the LQG/LTR method and the parameter robust LQG design.

In all configurations, the input u is an external force, and the output y is a displacement. Their state-space models are given as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{c}{m_1} & \frac{k}{m_1} & \frac{c}{m_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{k}{m_2} & \frac{c}{m_2} & -\frac{2k}{m_2} & -\frac{2c}{m_2} & \frac{k}{m_2} & \frac{c}{m_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{k}{m_3} & \frac{c}{m_3} & -\frac{2k}{m_3} & -\frac{2c}{m_3} & \frac{k}{m_3} & \frac{c}{m_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{k}{m_4} & \frac{c}{m_4} & -\frac{k}{m_4} & -\frac{c}{m_4} \end{bmatrix}$$

System N:

$$B_N = \left[0\ 0\ 0\ \frac{1}{m_2}\ 0\ 0\ 0\ 0\right]^T, \qquad C_N = [0\ 0\ 0\ 0\ 0\ 0\ 1\ 0]$$

System C: $B_C = B_N, \qquad C_C = [0\ 0\ 1\ 0\ 0\ 0\ 0\ 0]$

System E: $B_E = \left[0\ \frac{1}{m_1}\ 0\ 0\ 0\ 0\ 0\ 0\right]^T, \qquad C_E = C_N$

The numerical values for this study are chosen as follows;

$$m_1 = m_2 = m_3 = m_4 = 1$$

$$k = 4$$

$$c = 0.02$$

Suppose that each system is subject to a single parameter variation Δm_1 or Δm_2. In this example, Δm_1 is considered in systems N and C, and Δm_2 is considered in system E only. Note that these parameter variations produce a ΔA but not a ΔB. The input-output decompositions for Δm_1 and Δm_2 are given as

$$\hat{A} = A - \varepsilon M N$$

where, for Δm_1,

$$M = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]^T$$

$$N = [1\ 0.005\ -1\ -0.005\ 0\ 0\ 0\ 0]$$

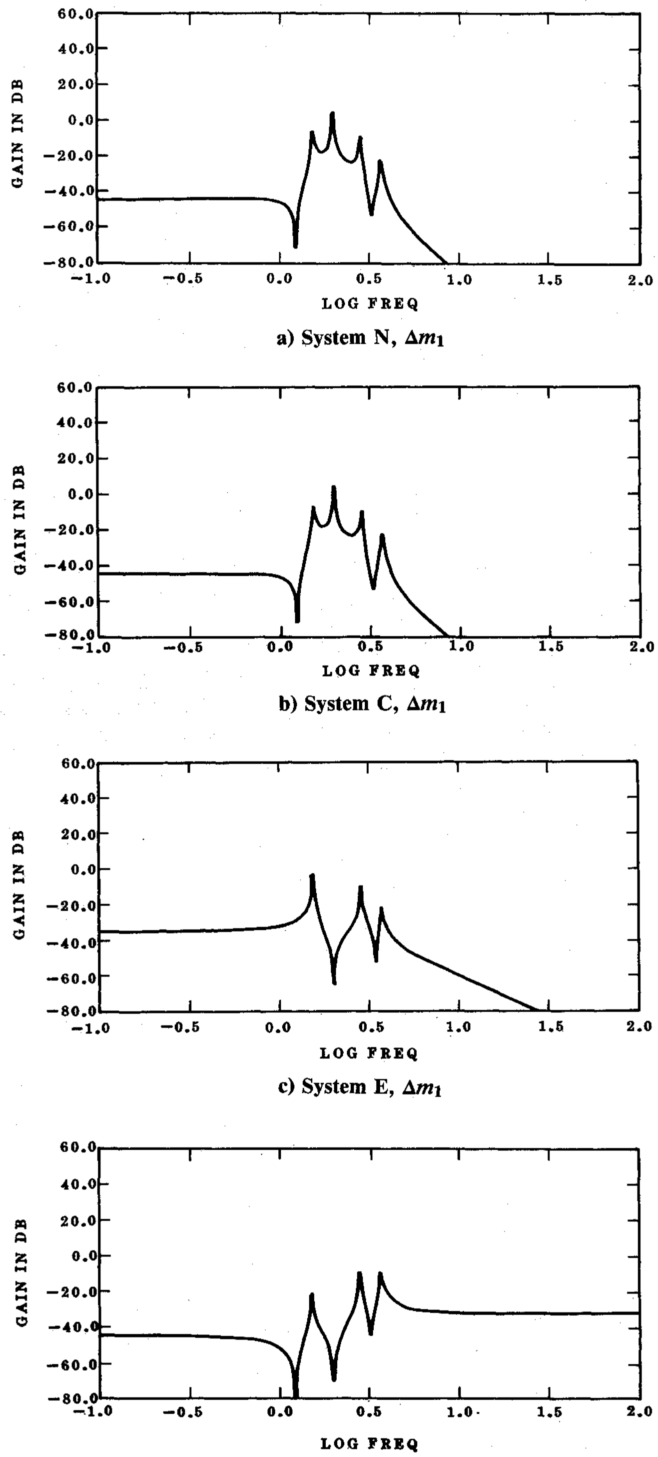


Fig. 3 Bode plots of $E(s)$.

and for Δm_2

$$M = [0\ 0\ 0\ 1\ 0\ 0\ 0\ 0]^T$$

$$N = [-1\ -0.005\ 2\ 0.01\ -1\ -0.005\ 0\ 0]$$

If we assume that all the masses are uncertain (multiple parameter uncertainties), then a state-space augmentation procedure is required to include ΔB.[3] The input-output decomposition in the augmented state-space representation is given as

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

$$N = \begin{bmatrix} 1 & 0.005 & -1 & -0.005 & 0 & 0 & 0 & 0 & \sigma \\ -1 & -0.005 & 2 & 0.01 & -1 & -0.005 & 0 & 0 & \tau \\ 0 & 0 & -1 & -0.005 & 2 & 0.01 & -1 & -0.005 & 0 \\ 0 & 0 & 0 & 0 & -1 & -0.005 & 1 & 0.005 & 0 \end{bmatrix}$$

where $\sigma = 0$ and $\tau = -0.25$ for systems C and N, and $\sigma = -0.25$ and $\tau = 0$ for system E.

The effect of parameter variations in lightly damped flexible structures is considered first. Parameter variations can be represented as loop uncertainties as follows. Let

$$\hat{G} = [I + E(s)]G(s)$$

where $G(s)$ and $\hat{G}(s)$ are the nominal plant and perturbed plant, respectively. Suppose that (A,B,C) and $(\hat{A},B,C)$ are the state-space realization of each plant, and that

$$\hat{A} = A - \varepsilon MN$$

Then, it is easy to show that

$$\hat{G}(s) = G(s) - \varepsilon C\phi M(I + \varepsilon N\phi M)^{-1}N\phi B$$

where $\phi = (sI - A)^{-1}$. Therefore, for SISO systems,

$$E(s) = -[\varepsilon C\phi M(I + \varepsilon N\phi M)^{-1}N\phi B]/C\phi B$$

For cases with a single uncertainty, the magnitude of $E(s)$ for $\varepsilon = 0.1$ is shown in Fig. 3 for each system. Figures 3a and 3b clearly show that a small parameter variation ($\varepsilon = 0.1$ corresponds to a 2.5% decrease in m_1) can cause significant loop variations. It is noted that a peak of +6 db occurs at the frequency of the zero that is affected by Δm_1. The two notches of Fig. 3b correspond to two other zeros that are not affected by Δm_1. From Fig. 3c, we can see that this tendency is much less significant for the end-to-end configuration, which has no zeros along the imaginary axis (the maximum $|E(s)|$ is −7 dB). Note that $E(s)$ is proportional to the inverse of $G(s)$. Therefore, if $G(s)$ has zeros very close to the imaginary axis, $E(s)$ will have a large value around the frequency of the zeros whenever these zeros are perturbed. This unique characteristic of flexible structures is, in fact, one of the main difficulties that hinders the direct application of modern frequency-domain methodology.

The LQG/LTR and the PRLQG are asymptotic methods. The difference is that the asymptotic LQG weightings are determined in different ways. In an LQG/LTR design, the input matrix B or the output matrix C is used, whereas the parameter robust LQG design uses M or N, which is associated with the input/output decomposition of ΔA. In our example, the filter is selected to produce asymptotic insensitivity to parameter variations. The weighting $\beta^2 BB^T$ is used for the LQG/LTR, and $\beta^2 MM^T$ for the parameter robust LQG design where $\beta = 100$ is chosen. Note that N is not used in this case. For this particular example, if M is given but N is arbitrary, then the parameter variations can be in any of the even rows of the matrix A. Thus, partial information on the structure of parameter variation (M in this case) may be enough for design purposes. This is not possible for the simultaneous asymptotic procedure or for an H^∞ optimization, which optimizes the infinity-norm of G_{22} because both M and N must be known. The regulator for both methods is designed with the cost

$$J = \int_0^\tau (100y^Ty + u^Tu)\, dt$$

Table 1 Stability range (system C)

	LQR/LTR	PRLQG, 1 parameter variation	PRLQG, 4 parameter variations
Δm_1	$-1.39 < \varepsilon < +0.14$	$-1.94 < \varepsilon < +4.00^a$	$-2.00^a < \varepsilon < +3.83$
Δm_2	—	—	$-200^a < \varepsilon < +4.00^a$
Δm_3	—	—	$-2.00^a < \varepsilon < +0.96$
Δm_4	—	—	$-200^a < \varepsilon < +1.76$

[a] $\varepsilon > +4.00$ or < -2.00 was not tested.

Table 2 Stability range (system N)

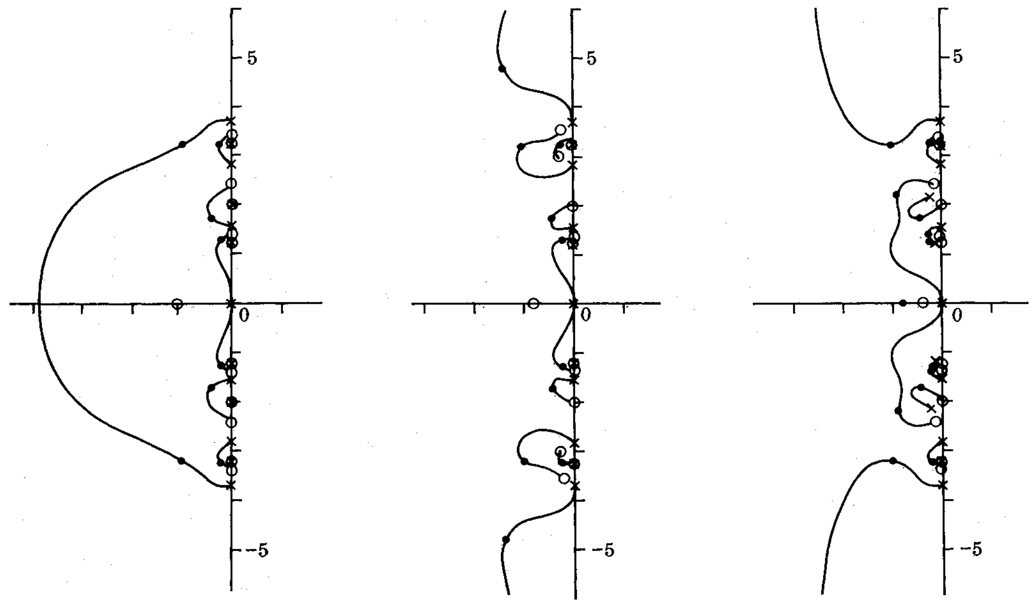
	LQG/LTR	PRLQG, 1 parameter variation	PRLQG, 4 parameter variations
Δm_1	$-0.22 < \varepsilon < +0.04$	$-0.81 < \varepsilon < +0.59$	$-1.20 < \varepsilon < +0.92$
Δm_2	—	—	$-1.35 < \varepsilon < +0.95$
Δm_3	—	—	$-1.98 < \varepsilon < +0.79$
Δm_4	—	—	$-2.00^a < \varepsilon < +2.10$

[a] $\varepsilon > +4.00$ or < -2.00 was not tested.

Table 3 Stability range (system E)

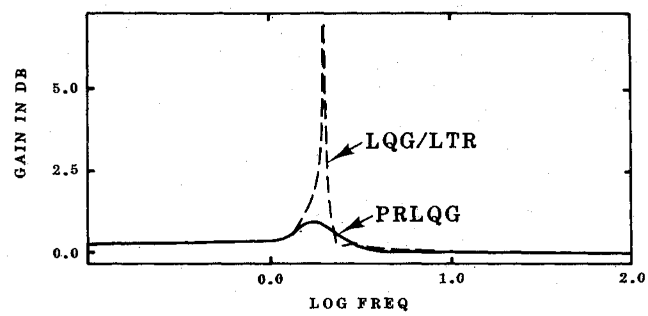
	LQG/LTR	PRLQG, 1 parameter variation	PRLQG, 4 parameter variations
Δm_1	—	—	$-1.69 < \varepsilon < +4.00^a$
Δm_2	$-0.30 < \varepsilon < +1.60$	$-0.60 < \varepsilon < +2.11$	$-2.00^a < \varepsilon < +4.00^a$
Δm_3	—	—	$-2.00^a < \varepsilon < +4.00^a$
Δm_4	—	—	$-2.00^a < \varepsilon < +3.80$

[a] $\varepsilon > +4.00$ or < -2.00 was not tested.

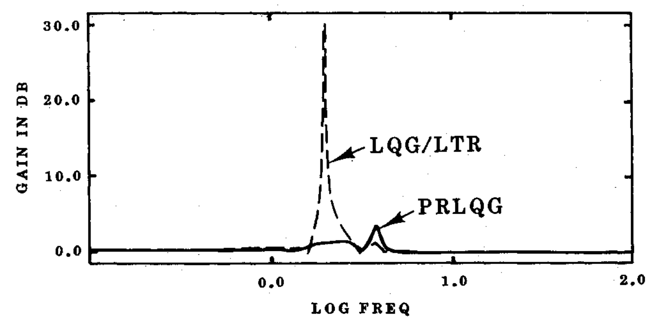


a) LQG/LTR b) PRLQG (1 p.v.) c) PRLQG (4 p.v.'s)

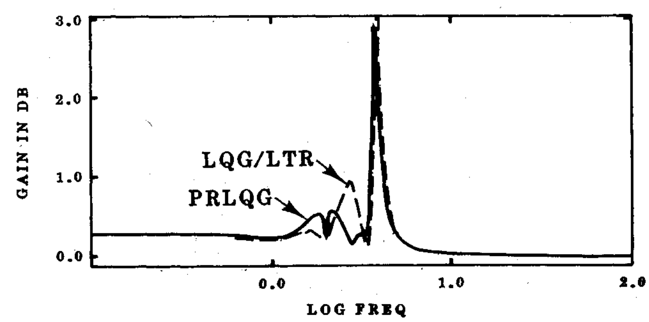
Fig. 4 Root loci for system C.



a) System C, Δm_1



b) System N, Δm_1



c) System E, Δm_2

Fig. 5 Bode plots of $G_{22}(s)$.

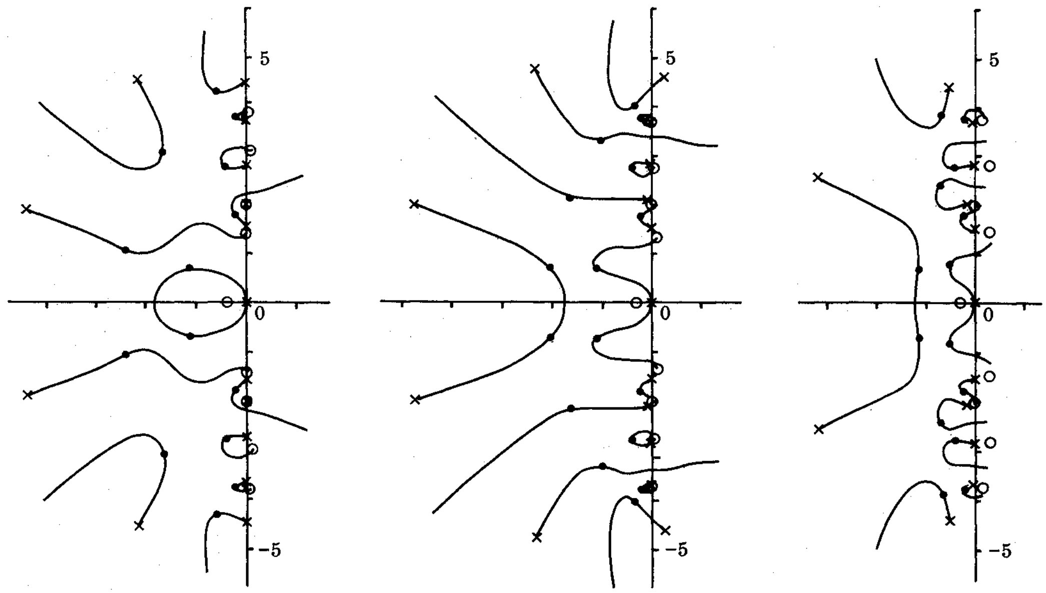
For convenience, the colocated system is discussed first. Any colocated system can be easily stabilized by a rate feedback, a rate + proportional feedback, or a simple lead-lag network. Since these classical methods result inherently in a robust system (i.e., stable for any plant uncertainty), the use of an optimal controller can be justified only when tight regulation is required.

Figure 4a shows the root loci for the LQG/LTR design. In this figure, we see that all the plant zeros are almost canceled by the compensator poles. Also, the compensator zeros are located right above each of the canceled plant zeros. In other words, each plant is effectively replaced by a compensator zero of a higher frequency. The root locus resembles that of a rate + proportional feedback, but the compensation is totally different.

The LQG/LTR asymptotically recovers the guaranteed stability margins of LQ regulator. However, this design method is usually very sensitive, even for colocated systems. As shown in Table 1, the LQG/LTR controller has poor robustness because the compensator pole that cancels the second plant zero becomes unstable with a small perturbation. The root locus for the parameter robust LQG (single parameter variations) is shown in Fig. 4b. We see that the first and the third zeros are canceled while the second zero remains untouched. It can be said that the parameter robust LQG avoids pole-zero cancellation, which can be a source of poor robustness. As shown in Table 1, the robustness is considerably increased when the PRLQG design is used. It is also very interesting that, for multiple parameter variations, the parameter robust LQG design results in a compensation very similar to the classical lead-lag compensation. The root locus shown in Fig. 4c clearly displays three pairs of near pole-zero cancellations of the compensator. The precise range of the tolerable ε for each design is given in Table 1. Furthermore, the improvement in robustness due to Δm_1 is dramatically displayed in the Bode plot of $G_{22}(s)$ shown in Fig. 5a.

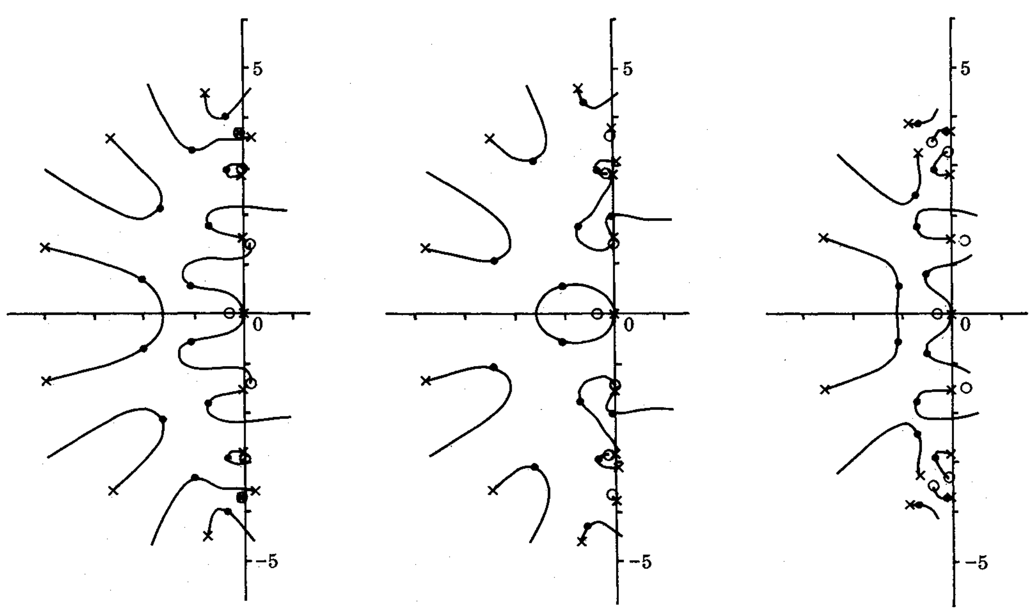
System N (Noncolocated System)

As shown in Fig. 2, the plant has a zero between the first and second flexible modes. As occurred in the colocated system, the LQG/LTR synthesis produces a compensator pole that cancels the plant zero. Figures 6a shows this situation. Again, this cancellation is the source of the robustness problem displayed in Table 2.



a) LQG/LTR b) PRLQG (1 p.v.) c) PRLQG (4 p.v.'s)

Fig. 6 Root loci for system N.



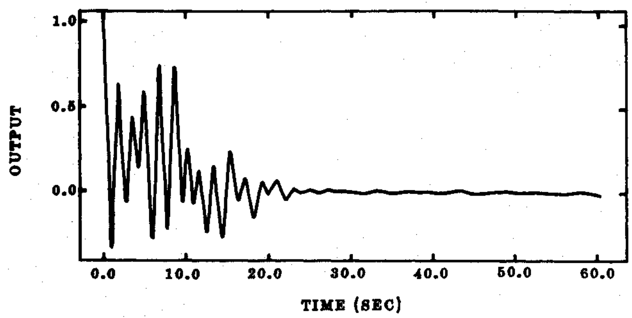
a) LQG/LTR b) PRLQG (1 p.v.) c) PRLQG (4 p.v.'s)

Fig. 7 Root loci for system E.

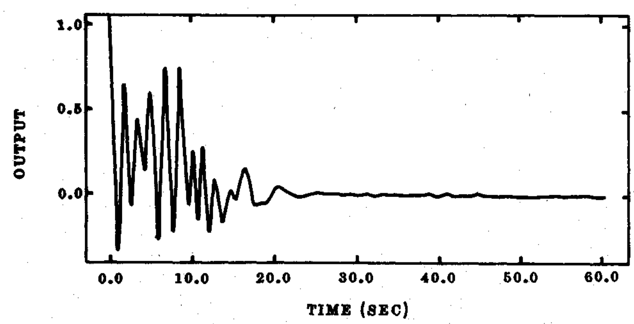
If the parameter robust LQG synthesis is employed, such cancellations do not occur, as shown in Figs. 6b and 6c. Also, the closed-loop system is considerably robust, as shown in Table 2. Figure 5b compared $G_{22}(s)$ using LQGL/LTR and parameter robust LQG synthesis. Robustness is significantly improved by using the parameter robust LQG design. The pole-zero patterns of the compensators are too complicated to interpret simply. Also, the parameter robust LQG design with multiple parameter variations (Fig. 6c) does not furnish a simple compensator as in the colocated case. Nevertheless, the pole-zero configuration of the compensator produces the proper phase characteristics to stabilize the system robustly. The range of ε for stability is given in Table 2 for the two synthesis methods. Furthermore, comparing Table 1 with Table 2 indicates that the robustness to parameter variation is harder to obtain for the noncolocated system.

System E (Noncolocated, End-to-end)

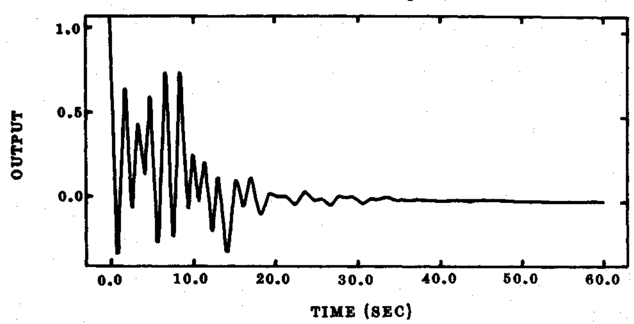
System E is a noncolocated system but has no transmission zeros. The two design methods produce non-minimum-phase and unstable compensation (Figs. 7a and 7b). For both methods, the closed-loop system is relatively robust to the parameter variation Δm_2, as shown in Table 3 and Fig. 5c. Table 3 also shows that the LQG/LTR gives somewhat satisfactory robustness compared to Tables 1 and 2. This is due to the lack of zeros in system E.



a) LQG/LTR

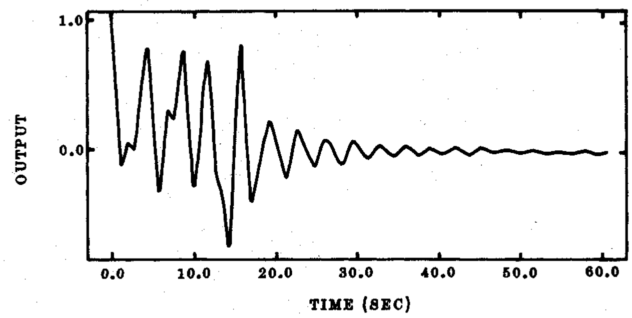


b) PRLQG (1 p.v.)

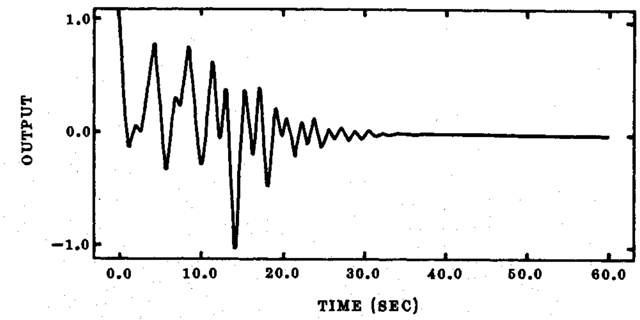


c) PRLQG (4 p.v.'s)

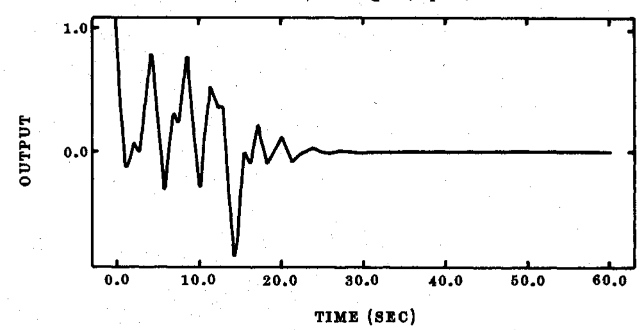
Fig. 8 Time response (system C).



a) LQG/LTR

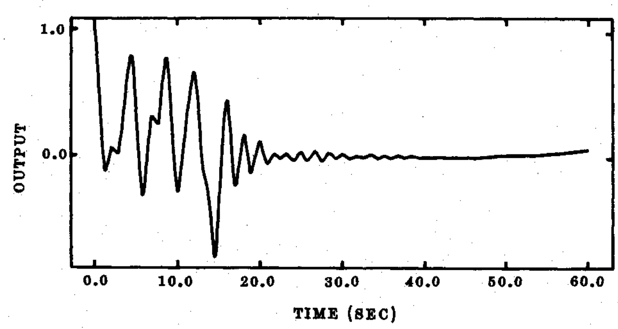


b) PRLQG (1 p.v.)

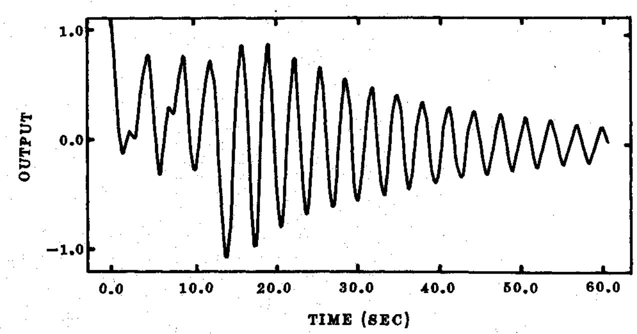


c) PRLQG (4 p.v.'s)

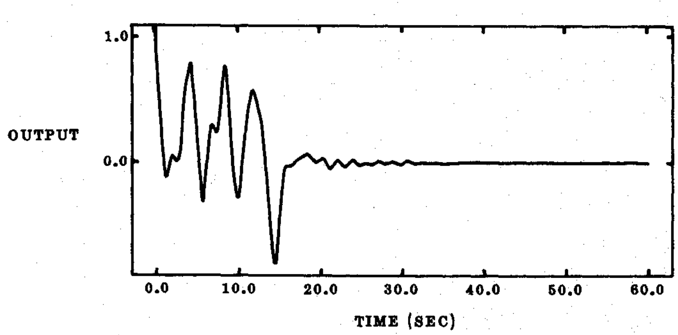
Fig. 9 Time response (system N).



a) LQG/LTR



b) PRLQG (1 p.v.)



c) PRLQG (4 p.v.'s)

Fig. 10 Time response (system E).

The time responses are given in Figs. 8–10 for cases in which the control is engaged after 10 s. For the parameter robust LQG design, the asymptotic poles of systems N and C are all canceled by the plant zeros so that they are not important for the closed-loop response. However, the asymptotic pole of system E (single uncertainty) causes some problem since it does not have a plant zero to cancel. In fact, this pole approaches the zero of $C\phi M$, which is not a plant zero. Therefore, the parameter robust LQG does not provide good time response, as displayed in Fig. 10b. In Fig. 7b, we can easily find this poorly damped mode. For multiple parameter variations, however, we obtain an acceptable time response (Fig. 10c) because there are no asymptotic finite poles around the imaginary axis. For all configurations, the parameter robust LQG design with multiple parameter variations provides satisfactory robustness without causing the time-response to deteriorate.

V. Conclusions

The parameter robust LQG synthesis procedure is shown to recover asymptotically the robustness matrix function of the LQ regulator or filter. The filter or regulator, respectively, is made insensitive to the parameter variations. It is seen that the model of the parameter variations, based on an internal-loop representation, is restricted by the dimension of the system outputs or inputs when the asymptotic recovery of the LQ regulator or filter, respectively is produced. Nevertheless, this procedure is seen to be a generalization of the LQG/LTR procedure and, within the limits of the theory, the best feature of both might be combined so that both parameter uncertainty and unmodeled dynamics can be addressed. To show the advantages of this generalization, the parameter robust LQG synthesis is applied to the control design of a flexible structure, and its performance is compared with that of the LQG/LTR method. The PRLQG synthesis provides satisfactory stability robustness and time response for noncolocated as well as colocated systems. The stability robustness problem associated with the LQG/LTR method was also pointed out. Although unmodeled dynamics was not considered, this example shows that if only unmodeled dynamics are considered, as is done in Ref. 11, stability may not be assured for small parameter variations.

Acknowledgments

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